Homework 9
MTH 829 Complex Analysis

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February 13, 2018

Proposition 0.1 (Exercise X.12.2). *Let $g(z) = z^7 + 4z^4 + z^3 + 1$. In the region $|z| < 1$, g has four zeroes and in the region $1 < |z| < 2$, f has three zeroes, counting multiplicity.*

Proof. Let $f(z) = z^7$. On the circle $|z| = 2$, we have

$$\begin{aligned} |f(z) - g(z)| &= |4z^4 + z^3 + 1| \leq |4z^4| + |z^3| + 1 = 4|z|^4 + |z|^3 + 1 = 73 \\ |f(z)| &= |2|^7 = 128 \end{aligned}$$

In particular, $|f(z) - g(z)| < |f(z)|$, so the hypotheses of Rouche's Theorem are satisfied, and we conclude that f and g have the same number of zeroes in the interior $|z| < 2$. Since we can see immediately that f has a zero of order 7 at zero and no others, this implies that g has seven zeroes in $|z| < 2$. Now let $h(z) = 4z^4$. On the circle $|z| = 1$, we have

$$\begin{aligned} |h(z) - g(z)| = |z^7 + z^3 + 1| \leq |z^7| + |z^3| + 1 = |z|^7 + |z|^3 + 1 &= 3 \\ |h(z)| = 4|z|^4 &= 4 \end{aligned}$$

Again we can apply Rouche's Theorem to conclude that g and h have the same number of zeroes in $|z| < 1$. Since h has a zero of order four at zero and no other zeroes, g has four zeroes in $|z| < 1$. Since g is degree seven, it has seven zeroes, counting multiplicity. Since four of the zeroes lie in $|z| < 1$, and all seven lie in $|z| < 2$, the other three must be in $1 < |z| < 2$. □

Proposition 0.2 (Exercise X.12.3). *For any $\epsilon > 0$, the function $g(z) = \sin z + \frac{1}{z+i}$ has infinitely many zeroes in the strip $|\operatorname{Im} z| < \epsilon$.*

Proof. Let $K_n = \overline{B}(2\pi n, \epsilon/2)$ be the closed ball of radius ϵ centered at $2\pi n$, for $n \in \mathbb{Z}$. Note that $K_n \subset \{z : |\operatorname{Im} z| < \epsilon\}$. Define $f(z) = \sin z$. Since f is 2π-periodic, we can compute the values of f on the boundary of K_n by considering f on the circle centered at zero. We may assume ϵ is small enough that the sets K_n are disjoint, so f does not vanish on $\partial K_n = \{\frac{\epsilon}{2} e^{it} : t \in [0, 2\pi]$ since the only zeroes of $\sin z$ are the centers of the K_n. Define

$$\delta = \min_{t \in [0,2\pi]} \left\{ \left| f\left(\frac{\epsilon}{2} e^{it} \right) \right| \right\}$$

We know δ exists since ∂K_n is compact, and $\delta > 0$ since f does not vanish on ∂K_n. Also on ∂K_n, we have

$$|f(z) - g(z)| = \left| \frac{1}{z+i} \right|$$

This goes to zero as $|z| \to \infty$, so we can choose N so that

$$|z| > N \implies \frac{1}{|z+i|} < \delta$$

Then on ∂K_n we have

$$|f(z) - g(z)| = \frac{1}{|z+i|} < \delta \leq |f(z)|$$

Note that f, g are holomorphic away from $-i$, so we can apply Rouche's Theorem to conclude that f, g have the same number of zeroes on the interior of K_n for $n \geq N$. Since f has one zero on the interior of every K_n, g has a zero on the interior of K_n for $n \geq N$. That is, g has infinitely many zeroes in $\bigcup_{n \in \mathbb{Z}} K_n$, which is a subset of $|\operatorname{Im} z| < \epsilon$. □

Proposition 0.3 (Exercise X.12.7)**.** *Let* $1 < a < \infty$ *and define* $g(z) = z + a - e^z$*. Then* g *has exactly one zero in the left half plane* $\operatorname{Re} z < 0$ *and it is on the real axis.*

Proof. Define $f(z) = z + a$. Let K_n be the rectangle with vertices $\pm nai, -2na \pm nai$ for $n \in \mathbb{N}$ with $n > a$. In the half plane $\operatorname{Re} z < 0$, we have $|e^z| \leq 1$ (with equality only on the imaginary axis). Thus on the boundary of K_n, we have

$$|f(z) - g(z)| = |e^z| \leq 1 \qquad |f(z)| = |z + a| \geq a > 1$$

The second inquality holds for $n \geq a > 1$, since then K_n contains the disk $|z - (-a)| < a$. Thus we can apply Rouche's Theorem, so f, g have the same number of zeroes in K_n. Clearly, f has one zero at $-a$, so g has one zero in K_n. Since n was arbitrary, g has only one zero in $\operatorname{Re} z < 0$. Finally, note that

$$g(-a) = -e^{-a} < 0 \qquad g(0) = a - 1 > 0$$

so by the intermediate value theorem, g has a zero in $(-a, 0)$ so its single zero is real.

□

Proposition 0.4 (Exercise X.12.8)**.** *Let* $0 < |a| < 1$ *and define* $g(z) = (z-1)^n e^z - a$*. Then* g *has exactly* n *roots, each of multiplicity one, in the half plane* $\operatorname{Re} z > 0$*. Furthermore, if* $|a| \leq 2^{-n}$*, then the roots all lie in the disk* $|z - 1| < \frac{1}{2}$*.*

Proof. Let $f(z) = (z-1)^n e^z$ and for $m \geq 2$ let K_m be the right half of the disk of radius m centered at zero. Then the closed disk of radius one centered at one is contained in K_m. Note that in the right half plane, $|e^z| \geq 1$, so on the boundary of K_m, we have

$$|f(z)| = |z-1|^n |e^z| \geq |z-1|^n \geq 1 > |a| = |f(z) - g(z)|$$

Thus by Rouche's Theorem, f, g have the same number of zeroes in K_m. Clearly f has a zero of multiplicity n at $z = 1$ and no other zeroes (e^z never vanishes). Thus g also has n

zeroes, and since m was arbitrary, g has n zeroes in the right half plane. We claim these zeroes all have multiplicity one. The derivative of g is

$$g'(z) = n(z-1)^n e^z + (z-1)^n e^z = (z-1)^{n-1} e^z (z - (1-n))$$

The only zeroes of $g'(z)$ occur at $z = 1$ and $z = 1 - n$. Pluggin in 1, we know it is not a zero of g, and $1 - n$ is not a zero of g in the right half plane. Since any zero of order 2 or more would have derivative zero at that point, all of g's zeroes in the right half plane have order one. Now suppose $|a| \leq 2^{-n}$, and let $K = \{z : |z - 1| < 1/2\}$. Then on ∂K,

$$|f(z) - g(z)| = |a| \leq 2^{-n} \leq 2^{-n}|e^z| = |f(z)|$$

so by Rouche's Theorem, the n zeroes of g are in K, since f has n zeroes in K. □

Proposition 0.5 (Exercise X.16.1)**.** *The following domains are all conformally equivalent to the open unit disk. In each case, we construct an explicit univalent holomorphic map from the open unit disk to the domain.*

1. *The upper half plane* $\operatorname{Im} z > 0$
2. *The whole plane with a slit along* $(-\infty, 0]$
3. *The strip* $0 < \operatorname{Im} z < 1$
4. *The first quadrant,* $\operatorname{Im} z > 0, \operatorname{Re} z > 0$
5. *The intersection of the unit disk with the upper half plane*
6. *The unit disk minus the segment* $[0, 1)$

Proof. Throughout, we use the fact that a composition of univalent functions is univalent (when the domain and image line up appropriately). (1) The map $\phi(z) = \frac{z-i}{iz-1}$ satisfies

$$\phi(i) = 0 \qquad \phi(1) = -1 \qquad \phi(-1) = 1 \qquad \phi(0) = i$$

The first three say that ϕ maps the unit circle to the real axis. Since $\phi(0) = i$, ϕ maps the interior of the unit circle (the open unit disk) to the side of the real axis containing i (the upper half plane). Since ϕ is a linear fractional transformation, it is univalent.

(2) Let ϕ be as above. We claim that the map $\psi(z) = (-i\phi(z))^2$ maps the open unit disk to the plane with slit $(-\infty, 0]$. As already noted, ϕ maps the open unit disk bijectively to the upper half plane. Multiplying by $-i$ rotates by an angle $-\pi/2$, so the upper half plane becomes the right half plane. The map $z \mapsto z^2$ takes the right half plane, which we can write as

$$\{re^{i\theta} : r > 0, \theta \in (-\pi/2, \pi/2)\}$$

and doubles the angle and squares the modulus to send it to

$$\{r^2 e^{i2\theta} : r > 0, \theta \in (-\pi/2, \pi/2)\} = \{re^{i\theta} : r > 0, \theta \in (-\pi, \pi)\} = \mathbb{C} \setminus \{-\infty, 0]$$

which is precisely the slit plane we wanted. Note that $z \to z^2$ is univalent on the right half plane, since everything has angle in $(-\pi/2, \pi/2)$, so doubling the angle doesn't introduce any overlaps in the image.

(3) Let ϕ be as above. We claim that the map $\alpha(z) = \frac{1}{\pi} \operatorname{Log} \phi(z)$ maps the open unit disk to the strip $0 < \operatorname{Im} z < 1$ univalently. As already noted, ϕ takes the open unit disk to the upper half plane. This region is simply connected, so there is a branch of $\log \phi$, and we just choose the principal branch. Note that

$$\operatorname{Log}(re^{i\theta}) = \ln |r| + i\theta$$

so applying Log to the region $\{re^{i\theta} : r > 0, \theta \in (0, \pi)\}$ gives the strip $0 < \operatorname{Im} z < \pi$, since $r \mapsto \ln |r|$ is onto all of $\mathbb{R}$. Finally, the dilation $z \mapsto \frac{1}{\pi} z$ shrinks the strip $0 < \operatorname{Im} z < \pi$ to the strip $0 < \operatorname{Im} z < 1$. These maps are all univalent, so the composition is univalent.

(4) Let ϕ be as above. We claim that the map $\beta(z) = \sqrt{\phi(z)}$ maps the open unit disk univalently to the first quadrant. We know that ϕ maps the open unit disk to the upper half plane, which is simply connected, so there is a branch of $\sqrt{\phi}$. Taking the principal branch, $\sqrt{z}$ takes the upper half plane and halves angles (and shrinks radii), so $\theta \in (0, \pi)$ gets shrunk to $\theta \in (0, \pi/2)$. Thus $\sqrt{z}$ takes the upper half plane to the first quadrant.

(5) Define the linear fractional transformation $f(z) = \frac{-z-1}{z-1}$. It satisfies

$$f(1) = \infty \qquad f(-i) = -i \qquad f(i) = i \qquad f(-1) = 0 \qquad f(-1+i) = -\frac{1}{5} + i\frac{2}{5}$$

so f maps the real axis onto the real axis and the imaginary axis onto the unit circle. Thus f maps each quadrant to one of four regions: the upper half of the unit disk, the lower half of the unit disk, the upper half plane minus the unit disk, or the lower half plane minus the unit disk. Since $f(-1+i) = -\frac{1}{5} + i\frac{2}{5}$, f maps the second quadrant to the upper half of the unit disk.

Then define $\eta(z) = f(i\beta(z))$, with β as above. We know that β maps the open unit disk to the first quadrant. Multiplication by i rotates the first quadrant to the second quadrant, and then η maps the second quadrant to the upper half of the unit disk. Thus η maps the open unit disk to the upper half of the unit disk, and η is a composition of univalent functions, so it is univalent.

(6) Let η be as above. Define $\chi(z) = \eta(z)^2$. We claim χ maps the open unit disk to the open unit disk minus the slit $[0, 1)$. The image of η is

$$\{re^{i\theta} : 0 < r < 1, 0 < \theta < \pi\}$$

so after applying the square map, this region becomes

$$\{r^2 e^{2i\theta} : 0 < r < 1, 0 < \theta < \pi\} = \{re^{i\theta} : 0 < r < 1, 0 < \theta < 2\pi\}$$

which is precisely the open unit disk minus the segment $[0, 1)$.

For the sake of concreteness, here are explicit formulas for $\phi, \psi, \alpha, \beta, \eta, \chi$.

$$\phi(z) = \frac{z-1}{iz-1} \qquad \psi(z) = \left((-i)\frac{z-1}{iz-1}\right)^2 \qquad \alpha(z) = \frac{1}{\pi} \operatorname{Log}\left(\frac{z-1}{iz-1}\right)$$

$$\beta(z) = \sqrt{\frac{z-i}{iz-1}} \qquad \eta(z) = \frac{-i\sqrt{\frac{z-i}{iz-1}} - 1}{i\sqrt{\frac{z-i}{iz-1}} - 1} \qquad \chi(z) = \left(\frac{-i\sqrt{\frac{z-i}{iz-1}} - 1}{i\sqrt{\frac{z-i}{iz-1}} - 1}\right)^2$$

□

Proposition 0.6 (Exercise X.16.5)**.** *The Koebe function, defined by*

$$f(z) = \frac{z}{(1-z)^2}$$

is univalent on the open unit disk. Its image is $\mathbb{C} \setminus (\infty, \frac{1}{4}]$.

Proof. Let ϕ be the linear fractional transformation

$$\phi(z) = \frac{1+z}{1-z}$$

Then $\phi(-1) = 0, \phi(i) = i$, and $\phi(-i) = -i$, so ϕ maps the unit circle to the imaginary axis. Also, $\phi(0) = 1$, so ϕ maps the open unit disk to the right half plane $\operatorname{Re} z > 0$. It is univalent since all LFTs are univalent. Now note that

$$\begin{aligned} \frac{1}{4}(\phi(z))^2 - \frac{1}{4} = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4} &= \frac{(1+z)^2 - (z-1)^2}{4(z-1)^2} \\ = \frac{(1+2z+z^2) - (1-2z+z^2)}{4(z-1)^2} &= \frac{4z}{4(z-1)^2} = \frac{z}{(1-z)^2} = f(z) \end{aligned}$$

So we have rewritten f as a composition of several simpler maps: ϕ, $z \mapsto z^2$, $z \mapsto \frac{1}{4}z$, and $z \mapsto z - \frac{1}{4}$. First, as we already showed, ϕ maps the unit disk univalently onto the right half plane. The map $z \mapsto z^2$ is univalent on the right half plane, since it doubles the argument. Its image is the slit plane $\mathbb{C} \setminus (-\infty, 0]$. The map $z \mapsto \frac{1}{4}z$ is a simple dilation, which is a bijection of the slit plane onto itself. Then the final map is a translation, which is univalent, and maps the slit plane $\mathbb{C} \setminus (-\infty, 0]$ to the shifted slit plane $\mathbb{C} \setminus (-\infty, \frac{1}{4}]$. A composition of univalent functions is univalent, so f is univalent on the open unit disk. □

Proposition 0.7 (Exercise X.19.3)**.** *Let G be a domain, and let*

$$A = \left\{ f : G \to \mathbb{C} \text{ holomorphic} \;\middle|\; \iint_G |f(x+iy)| dx\, dy \le 1 \right\}$$

Then A is a normal family.

Proof. We claim that A is locally uniformly bounded. If we can show this, then the Stieltjes-Osgood Theorem says that it is a normal family. Let $z_0 \in G$, and choose $r > 0$ so that $B(z_0, r) \subset G$. Let $f \in A$. Then f is holomorphic, so by the Mean Value Property (Exercise VII.6.1 of Sarason),

$$f(z_0) = \frac{1}{\pi(r/2)^2} \iint_{|z-z_0|<r/2} f(x+iy) dx\, dy$$

Since $f \in A$, the integral on the RHS is bounded by 1. Thus

$$f(z_0) \le \frac{4}{\pi r^2}$$

(Keep in mind that r depends on z_0.) By continuity of f at z_0, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \frac{4}{\pi r^2}$$

Now define $r' = \min(\delta, r)$. Then in $B(z_0, r')$, we have

$$|f(z)| \leq \frac{8}{\pi r^2}$$

(for all z in the ball). This process gives us a rule to assign to each $w \in G$ some $r_w > 0$ so that in $B(w, r_w)$ we have the inequality $|f(w)| \leq \frac{8}{\pi r_w^2}$. Let $K \subset G$ be compact. We can cover K by balls

$$K \subset \bigcup_{w \in K} B(w, r_w)$$

Since K is compact, there is a finite collection $w_1, \ldots, w_n$ so that

$$K \subset \bigcup_{i=1}^{n} B(w_i, r_{w_i})$$

Now let $r = \min\{r_{w_i}\}$. Since all the $r_w > 0$, we also have $r > 0$. Then on all of K, we have

$$|f(z)| \leq \frac{8}{\pi r^2}$$

That is, $\frac{8}{\pi r^2}$ is a bound for all of A on K. Thus A is locally uniformly bounded, so by the Stieljes-Osgood theorem it is a normal family. (Note: Sarason includes an additional hypothesis that G is bounded, which this proof shows to be unnecessary. However, if G is unbounded, then one can use the mean value property to show that A includes only the zero function, so the result is trivial in that case regardless.) ☐

Proposition 0.8 (Exercise X.19.4)**.** *Let $(f_n)_{n=1}^{\infty}$ be a locally uniformly bounded sequence of holomorphic functions on a domain G. Suppose that f_n converges pointwise on $A \subset G$ where A has a limit point in G. Then f_n converges locally uniformly on G.*

Proof. First, we claim that any two locally uniformly convergent subsequences of f_n must converge to the same limit. Suppose f_{n_k} and f_{m_k} are are locally uniformly convergent subsequences of f_n, with limits f, g respectively. By the Weierstrass convergence thoerem (VIII.15 of Sarason), f and g are holomorphic. Since f_n converges to f and g pointwise on A, f, g agree on A. Then by the identity principle, $f = g$. This proves the claim.

Now we prove by contradiction that f_n is locally uniformly Cauchy in G. Suppose it is not locally uniformly Cauchy. Then there exists a compact set $K \subset G$ and $\epsilon > 0$ and $z_0 \in K$ such that for every $N \in \mathbb{N}$, there exist $m_N, k_N > N$ so that

$$|f_{m_N}(z_0) - f_{k_N}(z_0)| > \epsilon$$

Then we can choose subsequences f_{m_n} and f_{k_n} of f_n so that

$$|f_{m_n}(z_0) - f_{k_n}(z_0)| > \epsilon \qquad \forall n \in \mathbb{N}$$

By the Stieljes-Osgood Theorem, there exist locally uniformly convergent subsequences of f_{m_n} and f_{k_n}, which we denote by $f^\ell_{m_n}$ and $f^p_{k_n}$. By the previous claim, $f^\ell_{m_n}$ and $f^p_{k_n}$ converge to the same limit. But this contradicts the fact that they can't converge pointwise to the same value at z_0, because there exists $\epsilon > 0$ so that

$$|f^\ell_{m_n}(z_0) - f^p_{k_n}(z_0)| > \epsilon \qquad \forall \ell, p$$

Thus f_n is locally uniformly Cauchy. We showed in class that being locally uinformly Cauchy is equivalent to being locally uniformly convergent. □